Journal of Sound and Vibration (1995) **187**(4), 695–712

A NON-LINEAR DYNAMIC MODEL FOR CABLES AND ITS APPLICATION TO A CABLE–STRUCTURE SYSTEM

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(*Received* 11 *May* 1992, *and in final form* 16 *January* 1995)

A set of governing equations for dynamic transverse motions of a cable with small sag is firstly obtained where effects of finite motions of the cable and small support motions are included. Cable motions are separated into two parts; quasi-static motions and modal motions. The quasi-static motions are the displacements of the cable which moves as an elastic tendon due to the support movements. The modal motions are expressed as a combination of the linear undamped modes of a cable with fixed ends. By Lagrange's equations of motion, the governing equations of the non-linear cable motions are obtained, where quadratic as well as cubic non-linear couplings appear. The cable model developed is next applied to a cable–structure system. A global/local mode approach is employed; the total motions are expressed in terms of global and local motions. The local motions are the modal motions of the cable, while the global motions are 3-D motions of the structure which include quasi-static motions of the cables only. The global are expressed as a combination of the eigenmodes computed by 3-D FEM in which cables are treated as tendons. By using Lagrange's formulation, algebraic governing equations are finally obtained in which global–local interaction appears as linear and quadratic couplings. The model for the system is modified to include the actuator motions at the cable supports; the actuator motions are in the cable axis direction. The study shows many possibilities for the control of global and/or local modes.

1. INTRODUCTION

Cables are very efficient structural members and hence have been widely used in many long-span structures, including cable-supported bridges, guyed towers and cable-supported roofs. Since cables are light, very flexible and lightly damped, structures utilizing cables, i.e., cable–structure systems, usually have various dynamic problems. Their modelling is therefore very important in predicting and controlling their response.

In common dynamic analysis practice, cable–structure systems are modelled by a 3D-finite element model in which the cables are usually simplified to equivalent tendon elements, and thus *global* vibration is obtained. *Local* vibration of the cable on the other hand, is obtained

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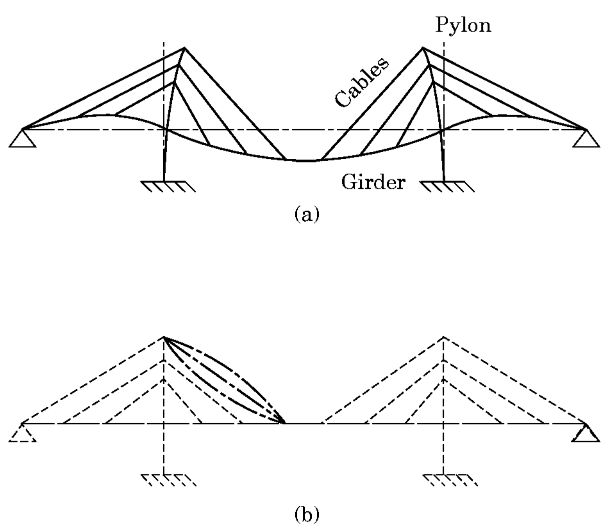
as vibration of a cable with fixed anchorages. Global and local vibrations are schematically shown in Figure 1, where a cable-stayed bridge is used as an example. The separate treatment of local and global vibrations ignores the interaction between them. Recent investigations by Maeda *et al*. [1], Kovacs [2], Abdel-Ghaffar and Khalifa [3] and Fujino *et al*. [4], however, show that interaction between local and global vibrations is significant; it appears as internal resonance, for example global motion can excite cable local motion resonantly, and *vice versa*.

Recently, Fujino *et al*. [4] conducted a dynamic experiment using a small cable-stayed beam model. They clearly showed the importance of the interaction of local/global motions; linear internal resonance as well as non-linear auto-parametric resonance. They proposed a global/local mode approach, which was used for the response prediction. It was found that the analytical model based on the global/local modes can explain the response characteristics qualitatively as well as quantitatively.

This study is a generalization of the model presented in reference [4] and presents a dynamic model for the global/local mode approach for cable-stayed bridges and some other types of cable–structure system. Firstly, the equation of motion for a cable with small sag is analytically derived, where small support motions as well as geometrical non-linearities are included. This is used for local cable vibration. Information on global modes are assumed to be available from conventional 3D-FEM analysis in which cables are treated as elastic tendons. Based on a Lagrange formulation, overall vibrations of the cable–structure system are expressed as generalized co-ordinates of the local cable modes and the global modes.

A 3D-FEM model for a cable–structure system can include the motion of cables by allocating degrees of freedom to each cable. However, the number of cables in a structure is often large and hence the numerical burden increases to an intolerable level even for highly developed digital computers. Although this local/global mode approach is approximate in a strict sense, there are many advantages: e.g., it needs a fewer number of degrees of freedom and the number of cables and number of local/global modes can be selected in the formulation; the formulation can easily accommodate cable non-linearities; since all coefficients of the present cable equations are explicit, physical linear/non-linear relations between local cable motions and global motions can be clearly seen.



(a)

(b)

Figure 1. Schematic drawings of (a) global vibration and (b) local vibration, for a cable-stayed bridge.

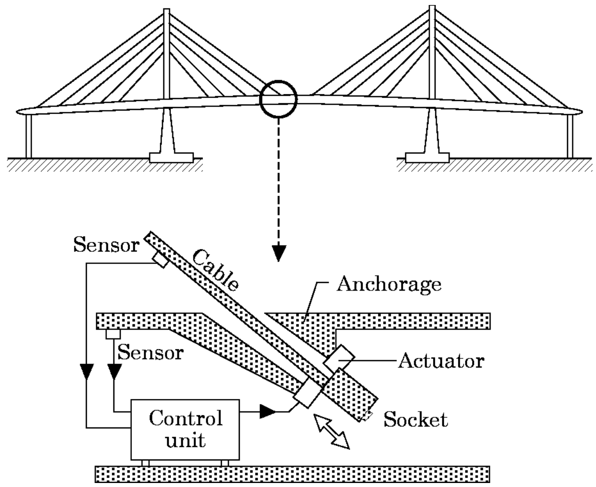


Figure 2. A cable–structure system with internal actuators.

The model is further extended to include the effects of the axial motion of internal actuators installed at cable anchorages (see Figure 2). These actuators are employed in an active control scheme by altering cable tension. This model enables one to make a systematic study on various control schemes for a cable–structure system.

2. CABLE WITH MOVING ANCHORAGES

In this section, motion of a single cable is investigated where the effects of global vibration are taken into account as motions at cable supports: i.e., anchorages. The following derivation is based on a uniform metallic cable which is the type normally used in engineering structures. The cable is highly stressed so that its weight to tension ratio is small. Its mass is uniformly distributed with density ρ. Axial stress is uniformly distributed over the cross-sectional area, A.

2.1. STATIC CONFIGURATION

In general, strain in a metallic cable is small because its modulus of elasticity is large. Thus, the static configuration of the cable can be assumed to be parabolic in the gravity plane [5]. For the co-ordinate system shown in Figure 3, it can be expressed as

$$w^{(s)} = \tfrac{1}{2}(\gamma L^2/\sigma^{(s)})[(x/L)-(x/L)^2] \qquad \text{and} \qquad u^{(s)} = v^{(s)} = 0, \tag{1, 2}$$

where u, v and w are axial, out-of-plane transverse and in-plane transverse displacements of the cable respectively; superscript (s) denotes the quantities referred to the static equilibrium state; L is the cable's chord length; x is the position measured along the cable's span; $\sigma^{(s)}$ is the static tensile stress; and γ is the component of distributed weight perpendicular to the chord line, which can be expressed as

$$\gamma = \rho g \cos\theta, \tag{3}$$

where g is the gravity constant (9·81 m/s^2) and θ is the angle of inclination measured from the horizontal line in the gravity plane.

2.2. PARTIAL DIFFERENTIAL EQUATIONS OF MOTION

The partial differential equations of cable motion are expressed in terms of static stress $\sigma^{(s)}$, dynamic stress σ, dynamical cable motions u, v and w, and external distributed forces X, Y and Z in the x, y and z directions, respectively [5].

For $0<x<L$,

$$\frac{\partial}{\partial x}\left\{(\sigma^{(s)}+\sigma)\frac{\partial u}{\partial x}+\sigma\right\}+X=\rho\frac{\partial^2 u}{\partial t^2}, \qquad \frac{\partial}{\partial x}\left\{(\sigma^{(s)}+\sigma)\frac{\partial v}{\partial x}\right\}+Y=\rho\frac{\partial^2 v}{\partial t^2}, \tag{4, 5}$$

$$\frac{\partial}{\partial x}\left\{(\sigma^{(s)}+\sigma)\frac{\partial w}{\partial x}+\sigma\frac{\partial w^{(s)}}{\partial x}\right\}+Z=\rho\frac{\partial^2 w}{\partial t^2}. \tag{6}$$

At supports a and b, the motions are induced from the global vibration and can be expressed as

$$u(x=0,t)=u_a(t), \qquad v(0,t)=v_a(t), \qquad w(0,t)=w_a(t), \tag{7}$$

$$u(x=L,t)=u_b(t), \qquad v(L,t)=v_b(t), \qquad w(L,t)=w_b(t). \tag{8}$$

They are assumed to be small: i.e.,

$$u_a, v_a, w_a, u_b, v_b, w_b \ll L. \tag{9}$$

Assume that stress in the cable is in its linear elastic range. Then the dynamic stress is expressed as a linear function of dynamic strain, ϵ,

$$\sigma(x,t)=E\epsilon(x,t), \tag{10}$$

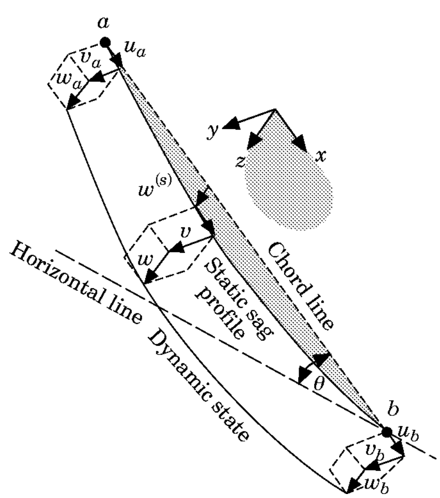


Figure 3. A cable with moving anchorages: co-ordinates and motions.

where the constant E is the cable modulus of elasticity. The non-linear strain–displacement relation is employed here due to the finite motion of the cable:

$$\epsilon(x, t) = \frac{\partial u}{\partial x} + \frac{\partial w^{(s)}}{\partial x}\frac{\partial w}{\partial x} + \tfrac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \tfrac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2. \tag{11}$$

Equations (4)–(11) completely describe the cable motions. However, further approximation is needed to obtain a reasonably simple model. It is known that the dynamic behaviour of a tightly stressed cable is close to that of a string. Thus, the fundamental transverse frequency is smaller than the fundamental frequency of the axial mode at least by one order of magnitude: i.e., in the quasi-static range of the axial mode [6]. Therefore the axial inertia force in equation (4) is very small and can be omitted. The axial distributed force X is assumed to be zero. From these assumptions and knowing that the axial strain is small, equation (4) can be simplified as

$$\partial\sigma/\partial x = 0: \tag{12}$$

i.e., the dynamic stress is constant over the span and consequently the dynamic strain is constant according to equation (10). Finally, a simplified set of governing equations for this study is obtained: equations (5)–(12).

2.3. QUASI-STATIC AND MODAL CABLE MOTIONS

The total time-dependent displacements are separated into two parts, quasi-static motions (denoted by superscript (q)) and modal motions (denoted by superscript (m)): i.e.,

$$u(x, t) = u^{(q)}(x, t) + u^{(m)}(x, t), \qquad v(x, t) = v^{(q)}(x, t) + v^{(m)}(x, t), \tag{13, 14}$$

$$w(x, t) = w^{(q)}(x, t) + w^{(m)}(x, t). \tag{15}$$

The quasi-static motions are the displacements of a cable which moves as an elastic tendon due to support movements, while the modal motions are expressed as a combination of modes of a cable with fixed ends.

2.3.1. *Quasi-static motions due to support movements*

By definition, quasi-static motions satisfy the time-dependent boundaries statistically. They are small and hence the linearized homogeneous form of equations (5), (6) and (12) are employed. The governing equations can be written as

$$\partial\sigma^{(q)}/\partial x = 0, \qquad \sigma^{(s)}\,\partial^2 v^{(q)}/\partial x^2 = 0, \qquad \sigma^{(s)}\,\partial^2 w^{(q)}/\partial x^2 + \sigma^{(q)}\,\partial^2 w^{(s)}/\partial x^2 = 0, \tag{16–18}$$

where the quasi-static stress, $\sigma^{(q)}$, can be obtained from equation (10) and the linearized form of equation (11):

$$\sigma^{(q)} = E\left(\frac{\partial u^{(q)}}{\partial x} + \frac{\partial w^{(s)}}{\partial x}\frac{\partial w^{(q)}}{\partial x}\right). \tag{19}$$

The boundary conditions are as follows:

$$u^{(q)}(0, t) = u_a(t), \qquad v^{(q)}(0, t) = v_a(t), \qquad w^{(q)}(0, t) = w_a(t), \tag{20}$$

$$u^{(q)}(L, t) = u_b(t), \qquad v^{(q)}(L, t) = v_b(t), \qquad w^{(q)}(L, t) = w_b(t). \tag{21}$$

The quasi-static motions can be analytically obtained from equations (16)–(21) as

$$u^{(q)} = u_a + \frac{E_q}{E}(u_b - u_a)\left(\frac{x}{L}\right) + \frac{\lambda^2}{4}\frac{E_q}{E}(u_b - u_a)\left[\left(\frac{x}{L}\right) - 2\left(\frac{x}{L}\right)^2 + \tfrac{4}{3}\left(\frac{x}{L}\right)^3\right]$$

$$-\tfrac{1}{2}(w_b - w_a)\left(\frac{\gamma L}{\sigma^{(s)}}\right)\left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right], \tag{22}$$

$$v^{(q)} = v_a + (v_b - v_a)x/L, \qquad w^{(q)} = w_a + (w_b - w_a)\frac{x}{L} - \tfrac{1}{2}\cdot\frac{\gamma L E_q}{\sigma^{(s)2}}\cdot(u_b - u_a)\left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right], \tag{23, 24}$$

where

$$E_q = \frac{1}{1 + \lambda^2/12}E \qquad \text{and} \qquad \lambda^2 = \frac{E}{\sigma^{(s)}}\left(\frac{\gamma L}{\sigma^{(s)}}\right)^2. \tag{25, 26}$$

The quasi-static stress can be obtained from equation (19) as

$$\sigma^{(q)} = E_q(u_b - u_a)/L. \tag{27}$$

The non-dimensionalized parameter λ^2, first introduced by Irvine [5], accounts for geometric and elastic effects. The effective axial modulus of the cable, E_q, is affected by this parameter.

It should be understood from equation (24) that the change of sag-profile is due to not only vertical but also longitudinal support motions.

2.3.2. *Modal motions*

The axial modal motion is small and is thus neglected:

$$u^{(m)}(x, t) \ll w^{(m)}(x, t), v^{(m)}(x, t). \tag{28}$$

For transverse modal motions, the separation-of-variables method is employed as

$$v^{(d)}(x, t) = \sum_n \phi_n(x) y_n(t), \qquad w^{(d)}(x, t) = \sum_n \psi_n(x) z_n(t), \tag{29, 30}$$

where out-of-plane and in-plane linear undamped mode shapes of a cable with fixed ends, ϕ_n and ψ_n, are selected for the spatial functions. The temporal coefficients, y_n and z_n, can be interpreted as generalized co-ordinates for out-of-plane and in-plane motions, respectively.

The linear undamped mode shapes of a cable with fixed ends can be obtained from the linearized homogeneous forms of equations (5), (6) and (12): i.e.,

$$\frac{\partial \sigma^{(m)}}{\partial x} = 0, \qquad \sigma^{(s)}\frac{\partial^2 v^{(m)}}{\partial x^2} = \rho\frac{\partial^2 v^{(m)}}{\partial t^2}, \qquad \sigma^{(s)}\frac{\partial^2 w^{(m)}}{\partial x^2} + \sigma^{(m)}\frac{\partial^2 w^{(s)}}{\partial x^2} = \rho\frac{\partial^2 w^{(m)}}{\partial t^2}, \tag{31–33}$$

where

$$\sigma^{(m)} = E\left(\frac{\partial u^{(m)}}{\partial x} + \frac{\partial w^{(s)}}{\partial x}\frac{\partial w^{(m)}}{\partial x}\right), \tag{34}$$

with the following boundary conditions:

$$u^{(m)}(0, t) = v^{(m)}(0, t) = w^{(m)}(0, t) = u^{(m)}(L, t) = v^{(m)}(L, t) = w^{(m)}(L, t) = 0. \tag{35}$$

These equations are the same as those obtained by Irvine [5]. The normal mode shapes for out-of-plane ($n = 1, 2, 3, \ldots$) and in-plane asymmetric modes ($n = 2, 4, 6, \ldots$) are

$$\phi_n = \psi_n = \sin(n\pi x/L), \tag{36}$$

with the following orthogonality conditions:

$$\int_0^L \phi_n \phi_m \, \mathrm{d}x = 0, \quad \int_0^L \frac{\mathrm{d}\phi_n}{\mathrm{d}x} \frac{\mathrm{d}\phi_m}{\mathrm{d}x} \, \mathrm{d}x = 0 \qquad \text{for } n \neq m, \tag{37a}$$

or

$$\int_0^L \psi_n \psi_m \, \mathrm{d}x = 0, \quad \int_0^L \frac{\mathrm{d}\psi_n}{\mathrm{d}x} \frac{\mathrm{d}\psi_m}{\mathrm{d}x} \, \mathrm{d}x = 0 \qquad \text{for } n \neq m. \tag{37b}$$

For in-plane symmetric modes ($n = 1, 3, 5, \ldots$), the normal mode shapes are

$$\psi_n = 1 - \cos(B_n x/L) - \tan(B_n/2) \sin(B_n x/L), \tag{38}$$

where the variable, B_n, has to be found from the following transcendental equation:

$$\tan(B_n/2) = (B_n/2) - (4/\lambda^2)(B_n/2)^3. \tag{39}$$

The orthogonality conditions in this case are

$$\int_0^L \psi_n \psi_m \, \mathrm{d}x = 0, \quad \int_0^L \frac{\mathrm{d}\psi_n}{\mathrm{d}x} \frac{\mathrm{d}\psi_m}{\mathrm{d}x} \, \mathrm{d}x + \frac{\lambda^2}{L^3} \left(\int_0^L \psi_n \, \mathrm{d}x \right) \left(\int_0^L \psi_m \, \mathrm{d}x \right) = 0 \qquad \text{for } n \neq m. \tag{40a, b}$$

Detailed derivations of the normal mode shapes and their orthogonality conditions are given in reference [7].

Up to this point, it has been established that equations (13)–(15) can be employed with complementary equations (22)–(24), (28)–(30) and (36)–(40).

2.4. LAGRANGE'S EQUATIONS OF MOTION

In this section, a dynamic model is obtained in terms of the local generalized co-ordinates (y_n and z_n). Lagrange formulation of the cable–structure model is employed for derivation. Lagrange's equations can be expressed as:

$$\frac{\partial}{\partial t}\left(\frac{\partial T_c}{\partial \dot{y}_n}\right) + \frac{\partial U_c}{\partial y_n} = \int_0^L YA \frac{\partial v}{\partial y_n} \, \mathrm{d}x, \qquad \frac{\partial}{\partial t}\left(\frac{\partial T_c}{\partial \dot{z}_n}\right) + \frac{\partial U_c}{\partial z_n} = \int_0^L ZA \frac{\partial w}{\partial z_n} \, \mathrm{d}x, \tag{41, 42}$$

where the kinetic potential, T_c, is defined by

$$T_c = \int_0^L \tfrac{1}{2} \rho A \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right] \mathrm{d}x \tag{43}$$

and the elastic potential, U_c, is defined by

$$U_c = \tfrac{1}{2}\,EAL\epsilon^2 + \tfrac{1}{2}\,\sigma^{(s)}A \int_0^L \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \mathrm{d}x, \tag{44}$$

where

$$\epsilon(t) = \frac{u_b - u_a}{L} + \frac{1}{L}\int_0^L \left\{\frac{\partial w^{(s)}}{\partial x}\cdot\frac{\partial w}{\partial x} + \tfrac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \tfrac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\} \mathrm{d}x. \tag{44$'$}$$

The first term on the right side of equation (44) is the potential due to elastic elongation of the cable while the second term is the potential due to static tensile stress [7, 8].

The time-varying strain is obtained by substituting the cable motions from equations (13)–(15) into equation (11). As the time-varying strain has been taken to be spatially uniform, equation (11) is then integrated along the cable axis with the boundary condition from equations (7) and (8). With the conditions

$$\left(\frac{u_b - u_a}{L}\right)^2, \left(\frac{v_b - v_a}{L}\right)^2, \left(\frac{w_b - w_a}{L}\right)^2 \ll \left(\frac{u_b - u_a}{L}\right) \qquad \text{and} \qquad \sigma^{(q)} \ll \sigma^{(s)}, \tag{45, 46}$$

the time-varying strain is finally obtained as

$$\epsilon = \frac{E_q}{E}\left(\frac{u_b - u_a}{L}\right) + \frac{\gamma}{\sigma^{(s)}L}\sum_n z_n \int_0^L \psi_n \,\mathrm{d}x + \sum_n \sum_m \frac{y_n y_m}{2L}\int_0^L \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\frac{\mathrm{d}\phi_m}{\mathrm{d}x}\,\mathrm{d}x$$

$$+ \sum_n \sum_m \frac{z_n z_m}{2L}\int_0^L \frac{\mathrm{d}\psi_n}{\mathrm{d}x}\frac{\mathrm{d}\psi_m}{\mathrm{d}x}\,\mathrm{d}x. \tag{47}$$

This equation together with equations (13)–(15) are substituted into the expressions for U_c and T_c. Then the orthogonality conditions (equations (37) and (40)) are first utilized in the expressions of U_c and T_c. Next, all integrations are carried out by using the mode shapes in equation (36), and $\psi_n(x) \approx \sin(n\pi x/L)$ for in-plane symmetric modes instead of equation (38). The approximation for in-plane symmetric modes is reasonable because the cable sag is small; more precisely, its non-dimensionalized parameter, λ^2, is less than one. Finally, the governing equations for the modal cable motions are obtained as follows:

For out-of-plane motion,

$$m_{yn}(\ddot{y}_n + 2\xi_{yn}\omega_{yn}\dot{y}_n + \omega_{yn}^2 y_n) + \sum_k \nu_{nk} y_n (y_k^2 + z_k^2) + \sum_k 2\beta_{nk} y_n z_k$$

$$+ 2\eta_n (u_b - u_a) y_n + \zeta_n(\ddot{v}_a + (-1)^{n+1}\ddot{v}_b) = F_{yn}; \tag{48}$$

For in-plane motion,

$$m_{zn}(\ddot{z}_n + 2\xi_{zn}\omega_{zn}\dot{z}_n + \omega_{zn}^2 z_n) + \sum_k \nu_{nk} z_n (y_k^2 + z_k^2) + \sum_k 2\beta_{nk} z_n z_k + \sum_k \beta_{kn}(y_k^2 + z_k^2)$$

$$+ 2\eta_n (u_b - u_a) z_n + \zeta_n(\ddot{w}_a + (-1)^{n+1}\ddot{w}_b) - \alpha_n(\ddot{u}_b - \ddot{u}_a) = F_{zn}, \qquad n = 1, 2, 3, \ldots. \tag{49}$$

Here

$$m_{yn}=m_{zn}=\tfrac{1}{2}\,\rho AL, \qquad m_{yn}\omega_{yn}^2=\frac{\sigma^{(s)}A\pi^2n^2}{2L}, \qquad m_{zn}\omega_{zn}^2=\frac{\sigma^{(s)}A\pi^2n^2}{2L}\,(1+k_n), \tag{50a–c}$$

$$k_n=(2\lambda^2/\pi^4n^4)(1+(-1)^{n+1})^2, \qquad v_{nk}=\frac{EA\pi^4n^2k^2}{8L^3}, \qquad \beta_{nk}=\frac{EA\pi\gamma n^2}{4L\sigma^{(s)}}\left(\frac{1+(-1)^{k+1}}{k}\right) \tag{50d–f}$$

$$\eta_n=\frac{E_qA\pi^2n^2}{4L^2}, \qquad \zeta_n=\frac{\rho AL}{n\pi}, \qquad \alpha_n=\frac{\rho AL}{n^3\pi^3}\,\frac{\gamma LE_q}{(\sigma^{(s)})^2}\,(1+(-1)^{n+1}), \tag{50g–i}$$

$$F_{yn}=\int_0^L YA\phi_n\,\mathrm{d}x, \qquad F_{zn}=\int_0^L ZA\psi_n\,\mathrm{d}x. \tag{50j, k}$$

In equations (48) and (49), proportional damping is assumed; the modal damping ratios ξ_{yn} and ξ_{zn} are introduced. The natural circular frequencies of the cable (ω_{yn} and ω_{zn}) can be explicitly expressed as follows: for out-of-plane modes,

$$\omega_{yn}=(n\pi/L)\sqrt{\sigma^{(s)}/\rho}\,, \tag{51}$$

for in-plane modes,

$$\omega_{zn}=n\pi/L\sqrt{(\sigma^{(s)}/\rho)(1+k_n)}. \tag{52}$$

The factor k_n is the effect of sag and defined in equation (50d). For even n, k_n is zero and for odd n, k_n is a positive value at most of the order of 10^{-2}.

The governing equations also reveal many possible phenomena such as the following: (a) internal non-linear couplings between the cable modes due to both quadratic and cubic non-linear terms exist and consequently internal resonance can be potentially induced with some frequency ratios: e.g., 1 : 1, 1 : 3, 1 : 2, 2 : 1 and 3 : 1 [6]; (b) support motions affect cable vibration in many ways; the axial support motion causes a variation of modal stiffness which might induce dynamic instability [2]; transverse support motions generate forces proportional to their accelerations; for in-plane symmetric modes, the axial support motion also generates a force proportional to its acceleration.

Non-linear vibrations of cables have been studied by many researchers. The stability of planar and non-planar undamped forced dynamic motions of a string was investigated by Miles [9]. Only one dominant mode in each plane was considered in his formulation. Effects of non-linearities on planar/non-planar dynamic motion of a sagged cable were investigated by Hagedorn *et al*. [10], Benedettini *et al*. [11] and Takahashi *et al*. [12, 13]. The normal in-plane symmetric modes of the sagged cable which were used as spatial functions in their formulations were numerically obtained and thus the coefficients in their equations are not in explicit forms.

Although the present model is limited in use to rather small sag, e.g. $\lambda^2<1$, all the coefficients in the governing equations are analytically expressed and non-linear phenomena can be studied with more physical insights. Moreover, the effects of support motions are included so that dynamic couplings in cable–structure systems can be investigated in what follows.

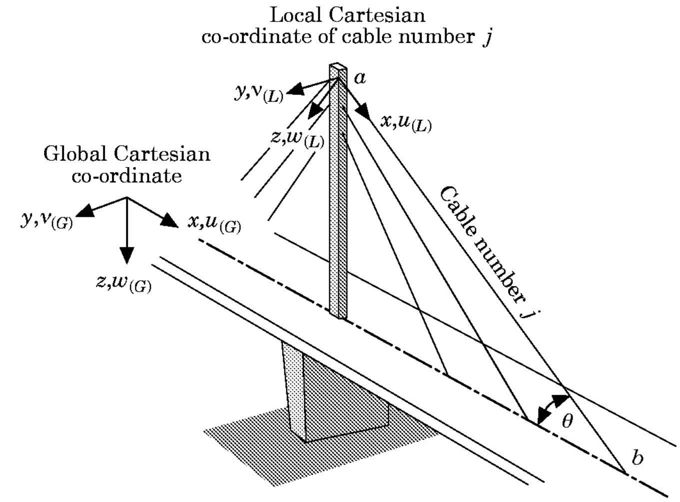


Figure 4. The local and global Cartesian co-ordinate systems.

3. NON-LINEAR MODEL OF CABLE–STRUCTURE SYSTEM

A non-linear model of cable–structure systems is proposed here by using the non-linear cable model obtained. The global/local mode approach [4, 7] is employed; total motions can be expressed as a sum of global and local motions. The global motions are 3-D motions of the structure which include quasi-static motions of cables. They can be expressed in terms of global generalized co-ordinates q_r as

$$\begin{Bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{Bmatrix}_{(G)} = \sum_r q_r(t) \begin{Bmatrix} \Phi_r^u(x,y,z) \\ \Phi_r^v(x,y,z) \\ \Phi_r^w(x,y,z) \end{Bmatrix}_{(G)}, \tag{53}$$

where Φ_r^u, Φ_r^v and Φ_r^w are x, y and z components of the rth global mode, respectively. Subscript (G) means that the quantity is referred to the *global Cartesian co-ordinates* shown in Figure 4. As stated, the global modes may be the eigenmodes computed by 3D-FEM where cables are treated as tendons in the formulation. The effect of initial stresses should be included in the 3D-FEM formulation.

The local motion is the motion of an individual cable that excludes the quasi-static motion:

$$\begin{Bmatrix} v(x,t) \\ w(x,t) \end{Bmatrix}_{(L,j)} = \begin{Bmatrix} \sum_n y_n(t)\phi_n(t) \\ \\ \sum_n z_n(t)\psi_n(t) \end{Bmatrix}_{(j)}. \tag{54}$$

Subscript (L) means that the quantity is referred to the *local Cartesian co-ordinates* shown in Figure 4 and subscript (j) refers to the jth cable. The finite local motions of cable are considered likewise to the previous section.

The motions at the cable supports, a and b, can be expressed in terms of the global generalized co-ordinates q_r as

$$\mathbf{u}_{(L,j)} = \begin{Bmatrix} u_b \\ v_b \\ w_b \\ u_a \\ v_a \\ w_a \end{Bmatrix}_{(L,j)} = \sum_r \mathbf{d}_{r(j)} q_r, \tag{55}$$

where

$$\mathbf{d}_{r(j)} = \begin{bmatrix} \mathbf{T}_m & 0 \\ 0 & \mathbf{T}_m \end{bmatrix}_{(j)} \cdot \begin{Bmatrix} \Phi_r^u(x_b, y_b, z_b) \\ \Phi_r^v(x_b, y_b, z_b) \\ \Phi_r^w(x_b, y_b, z_b) \\ \Phi_r^u(x_a, y_a, z_a) \\ \Phi_r^v(x_a, y_a, z_a) \\ \Phi_r^w(x_a, y_a, z_a) \end{Bmatrix}_{(G,j)}. \tag{56}$$

A 3×3 transformation matrix $\mathbf{T}_m$ is employed to transform a vector in the global to the local Cartesian co-ordinate. The matrix coefficients depend wholly on the angle between these two co-ordinate systems.

With these relations, elastic and kinetic potential of the structure can be expressed in terms of global and local generalized co-ordinates as

$$T = T_s(\dot{q}_r) + \sum_j T_{c(j)}(\dot{q}_r, \dot{y}_n, \dot{z}_n), \qquad U = U_s(q_r) + \sum_j U_{c(j)}(q_r, y_n, z_n), \tag{57, 58}$$

where subscripts s and c denote the structure and the cable, respectively. Employing Lagrange's equations of motion yields the following governing equations. For the kth global mode,

$$\frac{\partial}{\partial t}\left(\frac{\partial T_s}{\partial \dot{q}_k}\right) + \left(\frac{\partial U_s}{\partial q_k}\right) + \sum_j \frac{\partial}{\partial t}\left(\frac{\partial T_{c(j)}}{\partial \dot{q}_k}\right) + \sum_j \left(\frac{\partial U_{c(j)}}{\partial q_k}\right) = F_{qk}; \tag{59}$$

for the nth out-of-plane local mode of the jth cable,

$$\frac{\partial}{\partial t}\left(\frac{\partial T_c}{\partial \dot{y}_n}\right)_{(j)} + \left(\frac{\partial U_c}{\partial y_n}\right)_{(j)} = F_{yn(j)}; \tag{60}$$

for the nth in-plane local mode of the jth cable,

$$\frac{\partial}{\partial t}\left(\frac{\partial T_c}{\partial \dot{z}_n}\right)_{(j)} + \left(\frac{\partial U_c}{\partial z_n}\right)_{(j)} = F_{zn(j)}. \tag{61}$$

Here

$$F_{qk} = \iint_{S_{s,c}} \{p_x, p_y, p_z\}_{(G)} \cdot \frac{\partial}{\partial q_k} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}_{(G)} \mathrm{d}s = \iint_{S_{s,c}} \{p_x, p_y, p_z\}_{(G)} \cdot \begin{Bmatrix} \Phi_k^u \\ \Phi_k^v \\ \Phi_k^w \end{Bmatrix}_{(G)} \mathrm{d}s, \tag{62}$$

where p_x, p_y and p_z are the x, y and z components of the surface traction which is applied on the surface of the structure and the cable, $S_{S,C}$.

With the linearized finite displacement theory, the first two terms in equation (59) can be expressed as linear functions of the global generalized co-ordinates q_r:

$$\frac{\partial U_S}{\partial q_k}=\sum_r K_{skr}q_r, \qquad \frac{\partial}{\partial t}\left(\frac{\partial T_s}{\partial \dot{q}_k}\right)=\sum_r M_{skr}\ddot{q}_r. \tag{63, 64}$$

Here K_{skr} and M_{skr} are generalized global stiffness and mass, respectively.

The remaining two terms on the left side of equation (59) are associated with the cables. By using the chain rule and equation (55), the terms can be rewritten as

$$\frac{\partial U_{c(j)}}{\partial q_k}=\left\{\frac{\partial U_{c(j)}}{\partial \mathbf{u}_{(L,j)}}\right\}^{\mathrm{T}}\cdot\left\{\frac{\partial \mathbf{u}_{(L,j)}}{\partial q_k}\right\}=\mathbf{d}^{\mathrm{T}}_{k(j)}\cdot\left\{\frac{\partial U_{c(j)}}{\partial \mathbf{u}_{(L,j)}}\right\}, \tag{65}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T_{c(j)}}{\partial \dot{q}_k}\right)=\left\{\frac{\partial}{\partial t}\left(\frac{\partial T_{c(j)}}{\partial \dot{\mathbf{u}}_{(L,j)}}\right)\right\}^{\mathrm{T}}\cdot\left\{\frac{\partial \dot{\mathbf{u}}_{(L,j)}}{\partial \dot{q}_k}\right\}=\mathrm{d}^{\mathrm{T}}_{k(j)}\cdot\left\{\frac{\partial}{\partial t}\left(\frac{\partial T_{c(j)}}{\partial \dot{\mathbf{u}}_{(L,j)}}\right)\right\}. \tag{66}$$

These equations are in the same form as equation (62) and hence they can be referred to as generalized forces: i.e., products of force and derivative of motion. From this fact, the internal force induced by the cable, $\mathbf{F}_U$ and $\mathbf{F}_T$, can be defined as

$$\mathbf{F}_{U(L,j)}=\frac{\partial U_{c(j)}}{\partial \mathbf{u}_{(L,j)}}=(\mathbf{K}_c)_{(j)}\mathbf{u}_{(L,j)}+\left\{\sum_n \mathbf{Q}_n(y_n^2+z_n^2)\right\}_{(j)}, \tag{67}$$

$$\mathbf{F}_{T(L,j)}=\frac{\partial}{\partial t}\left(\frac{\partial T_{c(j)}}{\partial \dot{\mathbf{u}}_{(L,j)}}\right)=(\mathbf{M}_c)_{(j)}\ddot{\mathbf{u}}_{(L,j)}+\left\{\sum_n \mathbf{R}_n\ddot{y}_n+\mathbf{S}_n\ddot{z}_n\right\}_{(j)}, \tag{68}$$

where $\mathbf{K}_c$, $\mathbf{M}_c$, $\mathbf{Q}_n$, $\mathbf{R}_n$ and $\mathbf{S}_n$ are explicitly given in the Appendix.

These are, in fact, physical forces of cable number j, which act as concentrated forces on the structure. The forces are defined in local Cartesian co-ordinates and consist of two parts. The first terms in equations (67) and (68), respectively, are the elastic restoring force and the inertia force, respectively. They are induced by the quasi-static motions of the cable: i.e., global effects. The second terms are the effects of the local vibration of the cable. In equation (67), it can be seen that the local vibration generates non-linear forces due to finite motion of the cable and, in equation (68), it generates linear inertia forces.

It can be seen that the matrices $\mathbf{K}_c$ and $\mathbf{M}_c$ obtained here are different from those of the conventional tendon element. This depends on the magnitude of the cable static stress, i.e. sag. The difference is appreciable for large sag as shown in the Appendix. Thus, the matrices obtained are recommended for the calculation of the global modes.

Substituting equations (55), (67) and (68) into equations (65) and (66), the following equations are obtained:

$$\frac{\partial U_{c(j)}}{\partial q_k}=\sum_r (K_{ckr})_{(j)}q_r+\left\{\sum_n Q_{kn}(y_n^2+z_n^2)\right\}_{(j)}, \tag{69}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial T_{c(j)}}{\partial \dot{q}_k}\right)=\sum_r (M_{ckr})_{(j)}\ddot{q}_r+\left\{\sum_n (R_{kn}\ddot{y}_n+S_{kn}\ddot{z}_n)\right\}_{(j)}. \tag{70}$$

Here the coefficients for mode j are

$$K_{ckr}=\mathbf{d}_k^{\mathrm{T}}\mathbf{K}_c\mathbf{d}_r, \qquad M_{ckr}=\mathbf{d}_k^{\mathrm{T}}\mathbf{M}_c\mathbf{d}_r, \qquad Q_{kn}=\mathbf{d}_k^{\mathrm{T}}\mathbf{Q}_n, \qquad R_{kn}=\mathbf{d}_k^{\mathrm{T}}\mathbf{R}_n \quad \text{and} \quad S_{kn}=\mathbf{d}_k^{\mathrm{T}}\mathbf{S}_n. \tag{71a–e}$$

By substituting equations (63), (64), (69) and (70) into equation (59), the governing equation of the kth global mode can be obtained. For the global modes that possess orthogonal properties, it is

$$M_k[\ddot{q}_k+2\xi_k\omega_k\dot{q}_k+\omega_k^2q_k]+\sum_j\sum_n[R_{kn}\ddot{y}_n+S_{kn}\ddot{z}_n]_{(j)}+\sum_j\sum_n[Q_{kn}(y_n^2+z_n^2)]_{(j)}=F_{qk}. \tag{72}$$

In equation (72), proportional damping is assumed in the global modes; the modal damping ratio is ξ_k. M_k and ω_k are

$$M_k=M_{skr}+\sum_j(M_{ckr})_{(j)}, \qquad M_k\omega_k^2=K_{skr}+\sum_j(K_{ckr})_{(j)}. \tag{73, 74}$$

The governing equations for the local cable modes have already been derived in the previous section: i.e., equations (48) and (49). However, they are expressed in local co-ordinates. They can be transformed by using equations (55) and (71a)–(71e), and expressed as follows: for the nth out-of-plane local mode of the jth cable,

$$\left\{\begin{array}{c} m_{yn}(\ddot{y}_n+2\xi_{yn}\omega_{yn}\dot{y}_n+\omega_{yn}^2y_n)+\sum_k v_{nk}y_n(y_k^2+z_k^2) \\ +\sum_k 2\beta_{nk}y_nz_n+\sum_r 2Q_{rn}q_ry_n+\sum_r R_{rn}\ddot{q}_r \end{array}\right\}_{(j)}=F_{yn(j)}; \tag{75}$$

for the nth in-plane local mode of the jth cable,

$$\left\{\begin{array}{c} m_{zn}(\ddot{z}_n+2\xi_{zn}\omega_{zn}\dot{z}_n+\omega_{zn}^2z_n)+\sum_k v_{nk}z_n(y_k^2+z_k^2) \\ +\sum_k 2\beta_{nk}z_nz_k+\sum_k\beta_{kn}(y_k^2+z_k^2)+\sum_r 2Q_{rn}q_rz_n+\sum_r S_{rn}\ddot{q}_r \end{array}\right\}_{(j)}=F_{zn(j)}. \tag{76}$$

Equations (72), (75) and (76) are the complete non-linear dynamic model of the cable-structure system. Global–local interaction appears as linear and quadratic couplings. The topology of the coupling is present in Figure 5. The linear coupling will not exist if principal generalized co-ordinates are employed in the formulation of the model instead of the global and local ones. However, this requires tedious work in solving for the eigenmodes of a large matrix.

The dynamic interaction becomes significant only when the coupling generates internal resonance: i.e., when 1:1 and 2:1 frequency tunings are encountered in cases of linear and quadratic couplings respectively. It is possible for linear and quadratic couplings to occur simultaneously because the local modal frequencies are in harmonic ratios. Furthermore, the global–local couplings may also be combined with local–local coupling.

By investigating the frequency ratios, the modes the frequencies of which are linearly and non-linearly tuned can be identified and employed in equations (72), (75) and (76). Therefore the number of degrees of freedom to be solved is small.

4. CABLE–STRUCTURE SYSTEM WITH INTERNAL ACTUATORS

Vibration control of the cable–structure system is investigated in this section. Internal actuators which provide motions in the cable axial direction are employed at the cable anchorages (see Figure 2).

In order to obtain a model for the cable–structure system with the actuators, the expression of the axial support motion in equation (55) has to be redefined to include the displacement of the actuator ($\mathbf{u}_{c(L,j)}$): i.e.,

$$\mathbf{u}_{(L,j)} = \begin{Bmatrix} u_b \\ v_b \\ w_b \\ u_a \\ v_a \\ w_a \end{Bmatrix}_{(L,j)} + \begin{Bmatrix} u_{cb} \\ 0 \\ 0 \\ u_{ca} \\ 0 \\ 0 \end{Bmatrix}_{(L,j)} = \sum_r \mathbf{d}_{r(j)} q_r + \mathbf{u}_{c(L,j)}. \tag{77}$$

This modifies the expression of T_c and U_c in equations (57) and (58). The same procedure as for equations (59)–(68) is employed and the expressions for the internal forces $\mathbf{F}_{U(L,j)}$ and $\mathbf{F}_{T(L,j)}$ for this case are obtained as

$$\mathbf{F}_{U(L,j)} = \frac{\partial U_{c(j)}}{\partial \mathbf{u}_{(L,j)}} = (\mathbf{K}_c)_{(j)} \mathbf{u}_{(L,j)} + \left\{ \sum_n \mathbf{Q}_n (y_n^2 + z_n^2) \right\}_{(j)} + \mathbf{P}_A (u_{cb} - u_{ca}), \tag{78}$$

$$\mathbf{F}_{T(L,j)} = \mathbf{M}_{c(j)} \ddot{\mathbf{u}}_{(L,j)} + \left\{ \sum_n (\mathbf{R}_n \ddot{y}_n + \mathbf{S}_n \ddot{z}_n) \right\}_{(j)} + \mathbf{P}_B (\ddot{u}_{cb} - \ddot{u}_{ca}) + \mathbf{P}_C (\ddot{u}_{cb} + \ddot{u}_{ca}), \tag{79}$$

where the vectors $\mathbf{P}_A$, $\mathbf{P}_B$ and $\mathbf{P}_C$ are given in the Appendix. The terms associated with these coefficients are the effects of the control motion.

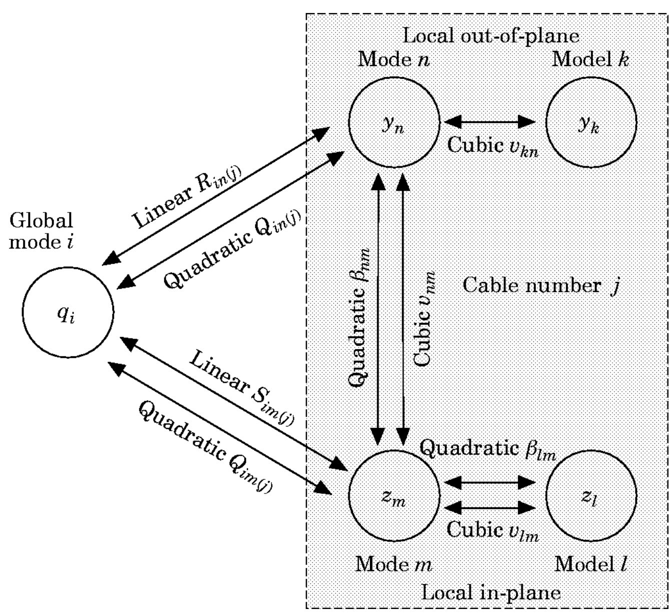


Figure 5. The topology of the global–local and local–local couplings in arbitrary chosen modes.

By following the same procedure as for equations (69)–(76), the following governing equations of the cable–structure system with internal actuators are obtained: for the kth global mode,

$$M_k[\ddot{q}_k+2\xi_k\omega_k\dot{q}_k+\omega_k^2 q_k]+\sum_j\sum_n[R_{kn}\ddot{y}_n+S_{kn}\ddot{z}_n]_{(j)}+\sum_j\sum_n[Q_{kn}(y_n^2+z_n^2)]_{(j)}$$

$$+\sum_j[P_{Ak}(u_{cb}-u_{ca})+P_{Bk}(\ddot{u}_{cb}-\ddot{u}_{ca})+P_{Ck}(\ddot{u}_{cb}+\ddot{u}_{ca})]_{(j)}=F_{qk}; \tag{80}$$

for the nth out-of-plane local mode of the jth cable,

$$\left\{\begin{array}{c} m_{yn}(\ddot{y}_n+2\xi_{yn}\omega_{yn}\dot{y}_n+\omega_{yn}^2 y_n)+\sum_k v_{nk}y_n(y_k^2+z_k^2) \\ +\sum_k 2\beta_{nk}y_n z_k+\sum_r 2Q_{rn}q_r y_n+\sum_r R_{rn}\ddot{q}_r+2\eta_n(u_{cb}-u_{ca})y_n \end{array}\right\}_{(j)}=F_{yn(j)}; \tag{81}$$

for the nth in-plane local mode of the jth cable,

$$\left\{\begin{array}{c} m_{zn}(\ddot{z}_n+2\xi_{zn}\omega_{zn}\dot{z}_n+\omega_{zn}^2 z_n)+\sum_k v_{nk}z_n(y_k^2+z_k^2) \\ +\sum_k 2\beta_{nk}z_n z_k+\sum_k \beta_{kn}(y_k^2+z_k^2)+\sum_r S_{rn}\ddot{q}_r+\sum_r 2Q_{rn}q_r z_n \\ +2\eta_n(u_{cb}-u_{ca})z_n-\alpha_n(\ddot{u}_{cb}-\ddot{u}_{ca}) \end{array}\right\}_{(j)}=F_{zn(j)}. \tag{82}$$

Here $P_{Ak}=\mathbf{d}_k^{\mathrm{T}}\mathbf{P}_A$, $P_{Bk}=d_k^{\mathrm{T}}\mathbf{P}_B$ and $P_{Ck}=\mathbf{d}_k^{\mathrm{T}}\mathbf{P}_C$.

The above equations are the governing equations of the cable–structure system (equations (72), (75) and (76)) with additional terms due to control effects. Many active control schemes can be established from these terms.

For the global modes, one has the following.

1. The *tendon force* ($P_{Ak}(u_{cb}-u_{ca})$) is the dynamic tension generated by the actuator movement. It is linearly proportional to the actuator motion and the well established linear control algorithm shown in references [14] and [15] can be employed. An experimental and analytical study on this control scheme with emphasis on the effects of internal resonance is presented in reference [16].

2. The *cable inertia forces* ($P_{Bk}(\ddot{u}_{cb}-\ddot{u}_{ca})$ and $P_{Ck}(\ddot{u}_{cb}+\ddot{u}_{ca})$) are linearly proportional to the actuator acceleration and the linear control algorithm can be employed.

For the local modes, one has the following.

1. *Variable stiffness terms* ($2\eta_n(u_{cb}-u_{ca})y_n$ and $2\eta_n(u_{cb}-u_{ca})z_n$) modify the transverse stiffness of the cable. It is possible for the parametric excitation phenomenon to occur [6]. With proper supply of the control-displacement signal, the cable vibration may be suppressed [17, 18]. The optimal condition for this scheme was obtained and experimentally verified [19].

2. *A sag-induced force in the local modes* ($\alpha_n(\ddot{u}_{cb}-\ddot{u}_{ca})$) exists in the symmetric in-plane modes of the cable only. It is a linear function of the control acceleration and the linear control algorithm can be applied [20].

Combination of these two effects can control local multi-modal response more effectively [21].

When these control schemes are used to suppress global modes as well as local modes, care should be taken since the control may introduce instability to the system by affecting the uncontrolled modes: i.e., control spillover. This occurs through the couplings of the system. Proper modes associated with these couplings have to be used in equations (80)–(82).

5. CONCLUSIONS

A dynamic model for transverse motions of a small-sag cable has been formulated. Finite cable motions and the cable support motions have been considered. The cable motion has been separated into quasi-static motion due to support movements and purely dynamic motion. This separation is suitable for later formulation of the cable–structure model. The modal motion can be treated as local motions, and global motions can be structure motions including the quasi-static motions of the cable. Due to this definition, the global modes can be obtained from the conventional 3D-FEM. The cable model shows that non-linear internal resonance can be induced due to quadratic and cubic non-linear terms. Interaction between global and local motions can be induced due to linear and quadratic couplings. From the knowledge that the interaction is significant only when internal resonance occurs, significant global and local modes can be selected and employed in the model. Thus, the number of degrees of freedom employed is small. The modified cable–structure model which includes actuators providing movements in the axial direction shows many possible schemes for active control. However, careful investigation is required to observe the effects of control spillover.

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APPENDIX: COEFFICIENTS OF MATRICES AND VECTORS

$\mathbf{K}_c$ is a 6×6 matrix with the following non-zero coefficients, $k_{l,m}$:

$$k_{1,1}=-k_{1,4}=-k_{4,1}=k_{4,4}=(A/L)(E_q+\sigma^{(s)}),$$
$$k_{2,2}=-k_{2,5}=k_{3,3}=-k_{3,6}=-k_{5,2}=k_{5,5}=-k_{6,3}=k_{6,6}=\sigma^{(s)}A/L. \tag{A1}$$

The ratio of $\sigma^{(s)}$ and E_q is the initial strain in the cable and is typically of the order of 10^{-3}. Thus, the terms associated with $\sigma^{(s)}$ are small and can be omitted.

$\mathbf{M}_c$ is a 6×6 matrix with the following non-zero coefficients, $m_{l,m}$:

$$m_{1,1}=m_{4,4}=\rho AL\left(\tfrac{1}{3}+\tfrac{1}{120}\left(\frac{\gamma LE_q}{\sigma^{(s)2}}\right)^2\right),\qquad m_{1,4}=m_{4,1}=\rho AL\left(\tfrac{1}{6}-\tfrac{1}{120}\left(\frac{\gamma LE_q}{\sigma^{(s)2}}\right)^2\right),$$

$$m_{2,2}=m_{5,5}=\frac{\rho AL}{3},\qquad m_{3,3}=m_{6,6}=\rho AL\left(\tfrac{1}{3}+\tfrac{1}{120}\left(\frac{\gamma L}{\sigma^{(s)}}\right)^2\right),$$

$$m_{2,5}=m_{5,2}=\frac{\rho AL}{6},\qquad m_{3,6}=m_{6,3}=\rho AL\left(\tfrac{1}{6}-\tfrac{1}{120}\left(\frac{\gamma L}{\sigma^{(s)}}\right)^2\right),$$

$$-m_{1,3}=-m_{3,1}=m_{4,6}=m_{6,4}=\frac{\rho AL}{24}\left(\frac{\gamma LE_q}{\sigma^{(s)2}}\right)\left(1+\frac{\sigma^{(s)}}{E_q}\right),$$

$$-m_{1,6}=-m_{6,1}=m_{3,4}=m_{4,3}=\frac{\rho AL}{24}\left(\frac{\gamma LE_q}{\sigma^{(s)2}}\right)\left(1-\frac{\sigma^{(s)}}{E_q}\right). \tag{A2}$$

For small sag, the value of $\gamma L/\sigma^{(s)}$ is small and the value related with this term in the above equation can be omitted. In this case, matrix $\mathbf{M}_c$ is the same as that of the tendon element. In case of large sag, this is not true; e.g., for a cable the length of which is 400 m, typical initial strain ($\sigma^{(s)}/E_q$) is 0·0025, weight to tension ratio ($\gamma L/\sigma^{(s)}$) is 0·05, and non-dimensionalized parameter λ^2 is 1, the factors $\frac{1}{120}(\gamma LE_q/\sigma^{(s)2})^2$ and $\frac{1}{24}(\gamma LE_q/\sigma^{(s)2})$ are about 3·3 and 0·83, respectively. These are significant and all terms in $\mathbf{M}_c$ are recommended for inclusion in this case.

$\mathbf{Q}_n$ is a 6×1 vector with the following non-zero coefficients, Q_l:

$$Q_1 = -Q_4 = E_q A\pi^2 n^2/4L^2. \tag{A3}$$

$\mathbf{R}_n$ is a 6×1 vector with the following non-zero coefficients, R_l:

$$R_2 = \{(-1)^{n+1}/n\}(\rho AL/\pi), \qquad R_5 = (1/n)(\rho AL/\pi). \tag{A4}$$

$\mathbf{S}_n$ is a 6×1 vector with the following non-zero coefficients, S_l:

$$-S_1 = S_4 = \frac{\rho AL}{\pi}\frac{\gamma LE_q}{\sigma^{(s)2}\pi^2}\left(\frac{1+(-1)^{n+1}}{n^3}\right), \qquad S_3 = \frac{(-1)^{n+1}}{n}\frac{\rho AL}{\pi}, \qquad S_6 = \frac{1}{n}\frac{\rho AL}{\pi}. \tag{A5}$$

$\mathbf{P}_A$ is a 6×1 vector with the following non-zero coefficients, P_{Al}:

$$P_{A1} = -P_{A4} = (E_q A/L) + (\sigma^{(s)}A/L). \tag{A6}$$

$\mathbf{P}_B$ is a 6×1 vector with the following non-zero coefficients, P_{Bl}:

$$P_{B1} = -P_{B4} = \tfrac{1}{12} + \tfrac{1}{120}(\gamma LE_q/\sigma^{(s)2})^2, \qquad P_{B3} = P_{B6} = -\tfrac{1}{24}(\gamma LE_q/\sigma^{(s)2}). \tag{A7}$$

$\mathbf{P}_C$ is a 6×1 vector with the following non-zero coefficients, P_{Cl}:

$$P_{C1} = P_{C4} = \tfrac{1}{4}, \qquad -P_{C3} = P_{C6} = \tfrac{1}{24}(\gamma L/\sigma^{(s)}). \tag{A8}$$